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EFFECT OF GAS EXPANSION ON SLUG LENGTH IN LONG PIPELINES

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Abstract—One of the important parameters in slug flow is the length of the liquid slug. Slug length is important in determining the average pressure drop as well as fluctuations in the pressure. Moreover, knowledge of the length of the slugs leaving long pipelines is crucial for the design of slug catchers. For short pipelines, the slug length is determined by the entrance phenomenon and by the stability of the slugs. For long pipelines the situation is not entirely clear. Long slugs may be formed due to terrain slugging. In this work, it is shown that long slugs can also be formed due to the decrease in pressure in the downstream direction.

INTRODUCTION

Slug length is one of the most important parameters in slug flow. Slug length was observed to be around 30 pipe diameters in horizontal pipes (Dukler & Hubbard 1975; Nicholson *et al.* 1978). For vertical pipes a value of 16 pipe diameters was quoted (Govier & Aziz 1972; Moissis & Griffith 1962; Akagawa & Sakaguchi 1960; Taitel *et al.* 1980). Slug length was found to be insensitive to flow conditions such as flow rates and fluid properties and to be within the general range of 10–40 pipe diameters. This fact was explained by Taitel *et al.* (1980), and recently by Brauner & Barnea (1986) and Dukler *et al.* (1985), on the basis that slug length is determined by the distance needed for the velocity profile to become fully developed. At the front of the slug the velocity profile has a transient character and it becomes fully developed towards the rear of the slug (the tail). Indeed, an analysis based on this concept of entry length results in slug lengths of the same order of magnitude as observed experimentally in small-diameter pipes.

Unfortunately, this is apparently not the whole story. Much longer slugs were observed for long and large-diameter pipelines (Brill *et al.* 1981). It seems that slugs continue to grow past the predicted maximum of $40D$. This is of major concern for applications that deal with long pipelines, primarily for the oil industry which uses long lines to transport oil and gas (Schmidt *et al.*, 1980, 1985; Taitel 1986). One of the reasons identified as a cause of the creation of long slugs was the terrain or severe slugging phenomenon. In this case, long slugs are generated at the lower spots of a long pipeline that has a downward slope followed by an upward slope. However, this phenomenon occurs only with low flow rates of liquid and gas, in which case the flow in the downward sections of the pipe is stratified. Thus, long slugs in a horizontal pipeline, for example, cannot be explained on the basis of the severe slugging phenomenon.

In this work we review and explain the different effects that determine slug length. In addition, a new phenomenon is identified that can explain and predict the growth of slugs in long pipelines which have a constant inclination angle and/or no sections with stratified flow. Admittedly, the verification of the proposed theory is not simple, because laboratory controlled experiments in long lines of constant inclination are not available. It is hoped that the present work will increase our understanding of slug flow in long pipelines.

ANALYSIS

Slug length is controlled by three separate phenomena: (1) entrance effects; (2) terrain geometry; and (3) slug stability.

As liquid and gas enter a pipeline and conditions are favorable for slug flow, slug flow is initiated by a cyclic phenomenon in which the pipe cross-sectional area is blocked by the liquid, resulting in competent bridging and the initiation of slugs. This phenomenon is controlled by the entrance effect and the frequency of slug generation per unit times depends on the cyclic nature of this process. Taitel & Dukler (1977) proposed a model which allows the calculation of the slug frequency, assuming that introduction of the two-phase mixture is in the form of stratified flow. The interface in the liquid film is unstable, waves grow and block the gas passage at which point a slug is formed. The slug scoops the liquid ahead of it and the next slug is generated only after the liquid film at the entrance is rebuilt. Slug length is inversely proportional to the slug frequency, the higher the slug frequency the shorter are the slugs that are generated. The slug frequency at the entrance, however, is not always the same as that along the pipe. In many cases the frequency of slug formation is very high and the frequency of the "fully developed slug flow" is determined by a sequence of merges that occur between slugs that increase the individual slug length and decrease their frequency. Slugs are termed stable when there is no growth in length as they travel downstream along the pipe. Taitel *et al.* (1980), for the case of upward flow, and Brauner & Barnea (1986) and Dukler *et al.* (1985), for the case of horizontal flow, suggested that a stable slug is a slug whose length is sufficiently long that the velocity profile at the "tail" of the liquid slug is fully developed and corresponds to the fully developed pipe flow. Dukler *et al.* (1985) assumed that the liquid in the slug front is well-mixed and that a boundary layer develops in the liquid slug until the flow becomes fully developed at the slug "tail". Brauner & Barnea (1986) considered a different mechanism. The liquid film that penetrates the liquid slug when the film is overrun by the liquid slug is considered as a wall jet. Transition from a wall jet flow into fully developed pipe flow towards the rear of the slug (the "tail") follows. Although the approaches of Brauner & Barnea (1986) and Dukler *et al.* (1985) are different, the end results for estimating slug length are similar and both analyses show that, indeed, the slug length is of the order of 16–30 pipe diameters. This seems to be correct for short lines. It does not explain, however, the existence of much longer slugs associated with flow in long pipelines.

A different phenomenon that controls slug length is terrain-induced slugging. This is also referred to as severe slugging (Schmidt *et al.* 1980, 1985; Taitel 1986). The severe slugging phenomenon has been considered primarily in connection with the off-shore riser system but it exists, of course, in any terrain geometry where a downsloping line is followed by an upward sloping line. When the flow rate of both liquid and gas is relatively low, the liquid accumulates in the low spots in the form of long liquid slugs. The gas upstream is trapped above the liquid slug and its pressure increases until it reaches a value that overcomes the hydrostatic head provided by the liquid and pushes the aforementioned liquid slug further downstream. Such a mechanism can provide quite long slugs which are definitely longer thant the "normal" 16–30 pipe diameters.

It seems, however, that even if the pipeline is perfectly horizontal, or has a constant slope, the liquid slugs tend to grow as they travel along the pipe. This is not explained by the simple stability criterion applied to slug flow, which resulted in a slug length of the order of $30D$. We propose here that the decrease in pressure along the pipe in the downstream direction is a mechanism which causes an increase in the slug length with distance and may result in the generation of long slugs in long tubes.

Slug Length in Short Pipelines

The process of growth or decay of "normal" slugs depends on the process of shedding from the rear of the slug and the pickup of the liquid in front of the slug. The pickup at the front of a slug, however, is essentially the amount shed from the rear of the preceding slug, i.e. the pickup is simply a collection of what is left behind by the previous slug. Shedding is believed to be caused (Dukler & Hubbard 1975; Gregory *et al.* 1978) by the velocity distribution at the rear of the liquid slug. The velocity of the liquid adjacent to the pipe wall is much lower than at the centerline. Consequently, slow moving liquid is left behind and liquid is lost from the slug body. As a result of this process, a liquid film stays behind and moves backward relative to the slug translational velocity, namely the velocity of the interface, V_t.

Thus, the rate of shedding determines the translational velocity. Figure 1 shows the velocity

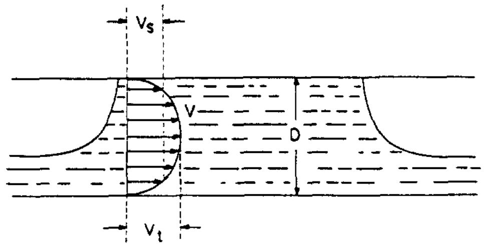

Figure 1. The shedding mechanism.

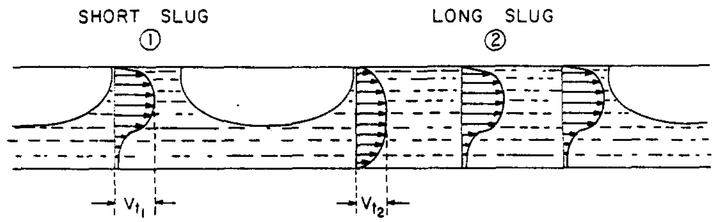

Figure 2. Velocity profiles in liquid slugs.

profile at the tail of a liquid slug. The mass of liquid per unit time left behing the slug is

$$X = \left(V_t A - \int V \mathrm{d}A \right) \rho_L R_s = A \rho_L R_s (V_t - V_s), \qquad [1]$$

where V is the local axial velocity, V_t is the maximum velocity of the velocity profile and V_s is the average velocity; A is the pipe cross-sectional area, ρ_L is the liquid density and R_s the liquid holdup; X is the liquid mass flow rate backward relative to a coordinate system that travels with translational velocity V_t.

Figure 2 shows two slugs. The front slug 2 is a long slug while slug 1 behind it is a short one. The velocity profile within the liquid slug is shown schematically as it develops from the mixing wall jet profile (the overrun liquid film can be considered as a wall jet into the slug) to a fully developed pipe flow at the rear of the slug. The first slug is however a short slug and its velocity profile at the slug "tail" is not yet fully developed. As a result, the maximum velocity in the velocity profile V_t is larger here and the shedding from slug 1 is larger than from slug 2. This means that slug 1 will lose liquid from the rear at a higher rate than it will be picking it up at the front. This will cause slug 1 to decay in length and eventually disappear and merge with the slug upstream of it (not shown in the figure). This is the process by which short slugs tend to disappear. This process however is terminated once all slugs are long enough for the velocity profile at the rear of the slugs to be fully developed. This process is usually the one which determines slug length in short tubes.

The term "usually" was used to exclude some of the cases where slug length is determined solely by the entrance phenomenon. If the generation of slugs at the entrance is of low frequency, the length of the generating slugs may be already higher than their minimum stable length. In this case, the length of the slugs will be determined by the frequency at the entrance. This is usually the situation for very low flow rates of liquid and gas near the transition boundary to stratified flow.

For a fully developed velocity profile, the ratio between the translational velocity V_t and the average velocity within the liquid slug V_s is approximately constant and can be given by the relation

$$V_t = (1 + C) V_s. \qquad [2]$$

For fully developed turbulent flow $C \approx 0.2$; namely $V_t / V_s = 1.2$, which is a typical ratio between the maximum and average velocity in a fully developed turbulent flow.

By using [2], the shedding ratio is given by

$$X = A \rho_L R_s C V_s. \qquad [3]$$

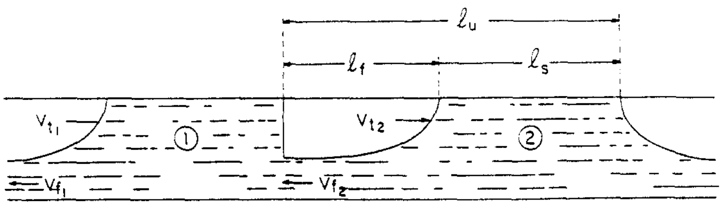

Figure 3. Fully developed slug flow.

Slug Length in Long Pipelines

In long pipelines there is a tendency for slugs to grow in length although they are aready fully developed, i.e. the velocity profile at the "tail" of the slug is a fully developed velocity profile. This growth is due to the decrease in pressure in the downstream direction and is analyzed below.

The average velocity in the cross section of a slug is given by

$$V_s = \frac{\dot{m}_L}{A\rho_L} + \frac{\dot{m}_G}{A\rho_G}. \quad [4]$$

where $\dot{m}_L$ is the liquid mass flow rate and $\dot{m}_G$ is the gas mass flow rate, both assumed to be constant. When the density of the liquid and the gas are constant, the average velocity is also constant at any cross section of the pipe. This simply reflects the fact that the volumetric flow rate is the sum of the liquid and gas flow rates and that it is constant for the case of constant densities.

When the pressure decreases in the downstream direction, the density of the gas decreases (the liquid density can usually be considered as constant) and, as a result, the average velocity V_s increases in the downstream direction. Assuming that the gas behaves as an ideal gas, [4] takes the form

$$V_s = \frac{\dot{m}_L}{A\rho_L} + \frac{\dot{m}_G \mathrm{R}T}{AP}, \quad [5]$$

where P is the pressure, T is the absolute temperature and R is the ideal gas constant. Since the shedding rate X is directly proportional to the average velocity V_s, [3], the shedding from slugs further downstream is larger than from those upstream. Consider now two consecutive slugs, as shown in figure 3. The shedding from the first slug is

$$X_1 = CA\rho_L R_s V_{s1}, \quad [6]$$

while the shedding from the second slug is

$$X_2 = CA\rho_L R_s V_{s2}; \quad [7]$$

where V_{s1} and V_{s2} are given by

$$V_{s1} = \frac{\dot{m}_L}{A\rho_L} + \frac{\dot{m}_G \mathrm{R}T}{AP} \quad [8]$$

and

$$V_{s2} = \frac{\dot{m}_L}{A\rho_L} + \frac{\dot{m}_G \mathrm{R}T}{A\left(P - \left|\frac{\mathrm{d}P}{\mathrm{d}x}\right| l_s\right)}, \quad [9]$$

where l_s is the liquid slug length and $|\mathrm{d}P/\mathrm{d}x|$ is the pressure drop in the liquid slug. Since the net liquid added to slug 1 is the difference between the pickup rate (which is equal to the shedding rate from slug 2) and the shedding rate, the net added liquid rate is

$$X_2 - X_1 = CA\rho_L R_s (V_{s2} - V_{s1}) = CA\rho_L R_s \frac{\dot{m}_G \mathrm{R}T}{AP^2}\left|\frac{\mathrm{d}P}{\mathrm{d}x}\right| l_s. \quad [10]$$

Simple mass balance on the liquid for slug 1 yields

$$X_2 - X_1 = \frac{dl_s}{dt} A\rho_L R_s. \qquad [11]$$

The axial position of a slug is given by

$$x = \int_0^t V_t \, dt = \int_0^t (1 + C)V_s dt, \qquad [12]$$

namely

$$\frac{dl_s}{dt} = \frac{dl_s}{dx}(1 + C)V_s. \qquad [13]$$

Using [10], [11] and [13] yields

$$\frac{dl_s}{dx} = \frac{C}{(1 + C)V_s} \frac{\dot{m}_G RT}{AP^2} \left|\frac{dP}{dx}\right| l_s. \qquad [14]$$

Equation [14] yields the growth of the liquid slug length with the axial coordinate x. The length of a slug unit l_u is the sum of the liquid slug length l_s and the film zone length l_f and can be calculated from a mass balance on the liquid as follows:

$$\dot{m}_L = \rho_L R_s A V_s \frac{l_s}{l_u} - \rho_L R_f A V_f \frac{l_f}{l_u}, \qquad [15]$$

where R_f and V_f are the liquid holdup and the liquid velocity of the film.

Equation [15] can be solved for the slug unit length, l_u, provided that V_f and R_f are known. The calculation of these parameters may be somewhat cumbersome if the exact variation of the film thickness, or the film local holdup R_f, is considered (Dukler & Hubbard 1975). As a good approximation one can consider the film thickness as a constant equal to the equilibrium film thickness, namely the film thickness far away from the liquid slug ahead of the liquid film. Note also that for vertical slug flow this assumption has been used extensively (Fernandes *et al.* 1983; Taitel *et al.* 1980). The solution of this equilibrium holdup, R_f, and the equilibrium film velocity, V_f, is detailed in the appendix. Note that the inclination angle, β, is considered positive for upwards inclination and V_f in this case is in the backward direction and is considered positive. For the case of horizontal flow the equilibrium velocity is obviously zero and the solution is considerably simplified. Using [15] the expression for the slug unit length is

$$l_u = \frac{R_s V_s + R_f V_f}{\dot{m}_L + \rho_L A R_f V_f} A\rho_L l_s. \qquad [16]$$

Equations [14] and [16] can now be used to predict the slug length with the axial position. A simple way of intergrating these equations numerically is to use the slug unit length as the axial discretization distance, Δx. In this case,

$$l_{s,n+1} = l_{sn} + l_{un}\left(\frac{C}{1 + C}\right)\left(\frac{\dot{m}_G RT}{AV_{sn}P_n^2}\right)\left|\frac{dP}{dx}\right|_n l_{sn}, \qquad [17]$$

where n is a digital count of the slug number. Likewise,

$$P_{n+1} = P_n - \left|\frac{dP}{dx}\right|_n l_{sn}. \qquad [18]$$

Once the pressure at the next slug unit is known all variables at the position $n + 1$ are calculated and so on.

Two additional inputs are needed for the aforementioned calculation. The liquid holdup within the liquid slug and the pressure drop in the liquid slug. The latter is calculated from

$$\left|\frac{dP}{dx}\right| = \left(\frac{4}{D}\right) f\rho_s \frac{V_s^2}{2} + \frac{(V_s + V_f)(V_t - V_s)\rho_L R_s}{l_s} + \rho_s g \sin\beta, \qquad [19]$$

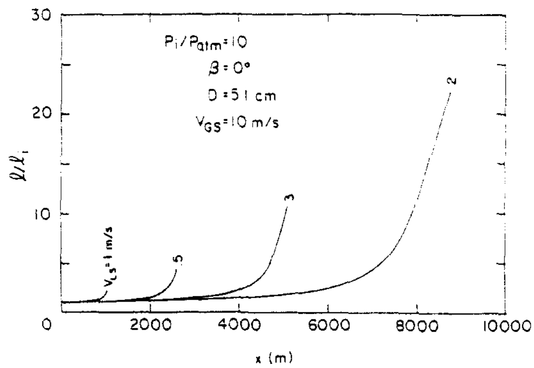

Figure 4. Slug length for horizontal flow.

where the first term on the r.h.s. is the frictional pressure drop, the second term is the acceleration pressure drop that results from the force necessary to accelerate the liquid film at the slug front to the slug velocity in the pickup process (Dukler & Hubbard 1975) and the third term is the gravitational pressure drop; f is the friction factor which depends on the Reynolds number and the pipe roughness. In this work we used a constant value for $f = 0.005$, for simplicity.

The liquid holdup in the slug was shown to depend on the liquid velocity V_s (Barnea & Brauner 1984; Gregory *et al.* 1978). In this work the simpler expression of Gregory *et al.* was used, namely

$$R_s = \frac{1}{1 + (0.115 V_s)^{1.39}}, \qquad [20]$$

where V_s is given in m/s. For very high velocity values the limiting value of $R_s = 0.48$ was assumed, since for lower values of liquid holdup it is not possible to form a competent blockage for gas passage (Brauner & Barnea 1986).

RESULTS AND DISCUSSION

The results of the calculations are demonstrated for a water–air system in a 5.1 cm dia pipe.

Figure 4 shows the relative liquid slug length, namely the liquid slug length divided by its length at the entrance (which is assumed to be $30D$), as a function of the distance for the case where the inlet pressure is 10 times atmospheric pressure. The flow rate of the gas, in terms of the superficial velocity at atmospheric conditions, is 10 m/s. The curves show the growth of the slugs with the axial distance, x, up to the point where the pressure equals atmospheric pressure. As seen, the growth of the slug length depends very strongly on the liquid flow rate. For a liquid superficial velocity of 1 m/s the increase in slug length is only 2-fold in a distance of 1000 m. For a liquid flow rate of 0.2 m/s the amplification of the liquid slug length is more than 20-fold but this occurs only in a very long pipe (about 9 km). It is obvious that for lower liquid flow rates the distance to reach 1 atm is much larger than for higher liquid flow rates. Yet, the strong effect on the final amplification ratio is not easily foreseen.

Another observation is that the increase in liquid slug length is very low near the entrance, but that slug length increases exponentially near the exit. This is due to the rapid decrease in the gas density as the pressure decreases near the exit. Upstream, the pressure is high and the gas behaves more like an incompressible fluid, in which case there is no growth in length.

Figure 5 shows the effect of upward inclination on the amplification ratio. As seen, the effect of inclination is small but the length of the pipe at which this amplification occurs is much shorter. This is due to the more rapid pressure drop in upward flow which is caused by the gravitational pressure drop and also to the increase in the acceleration pressure drop.

Figure 6 shows the effect of reducing the inlet pressure. Again, inlet pressure has a minor influence on the amplification ratio, but, similar to the case of upward inclination, slug growth occurs in shorter tubes.

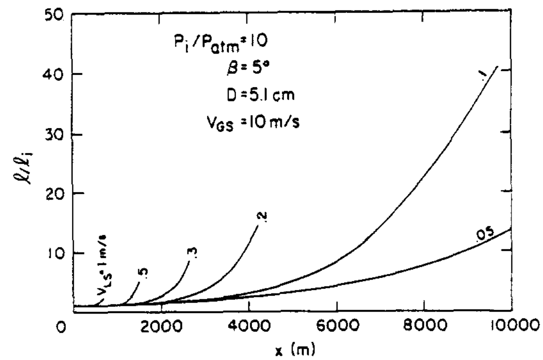

Figure 5. Slug length: effect of upward inclination.

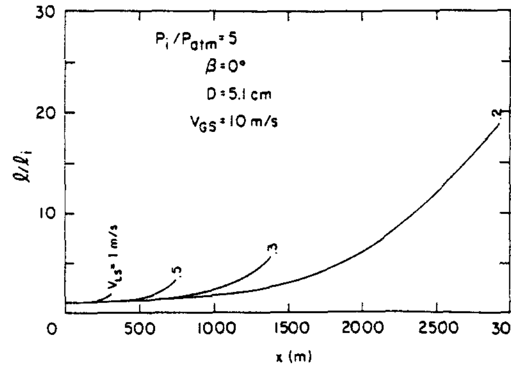

Figure 6. Slug length: effect of inlet pressure.

Figure 7 shows the effect of gas flow rate on the amplification ratio. Decreasing the gas flow rate by a factor of 2 decreases the amplification ratio considerably and, at the same time, longer pipes are needed for the pressure to reduce to atmospheric pressure.

Finally, figure 8 shows the effect of pipe diameter. Comparison between figures 8 and 4 shows that the effect of pipe diameter on the amplification ratio is negligible and that the growth rate of the slug length is the same in terms of the normalized distance x/D.

It is somewhat suprising to notice the large effect of the liquid and gas flow rates on the amplification ratio. Slug length increases considerably as the liquid flow rate decreases and the gas flow rate increases. The effects of inlet pressure, angle of inclination as well as pipe diameter were found to be small.

It would have been interesting to compare the present results with experimental data. Unfortunately this is very difficult since it requires a very long pipe (of the order of few kilometers for a 5.1 cm dia pipe), which is clearly unpractical. The experimental field data of Brill *et al.* (1981), although the prime motivation for this study, turns out not to be associated with the present phenomenon. Observation of the test results shows that in the Prudhoe Bay experiment the ratio between inlet and outlet presure was of the order of unity and that the absolute pressure varied in a relative narrow range (40–60 atm). Therefore, the mechanism proposed here is not the cause of the long slugs in the Prudhoe Bay experiment. The long slugs observed in the Prudhoe Bay experiment thus remain unexplained and are currently under investigation.

SUMMARY AND CONCLUSIONS

Gas expansion due to the decrease in pressure in long pipelines has been shown to cause slug length growth in two-phase gas–liquid flow.

It was found that the effect of pressure variation on the length of slugs is important for low liquid flow rate and high gas flow rate and that the effects of inlet pressure, pipe diameter and angle of

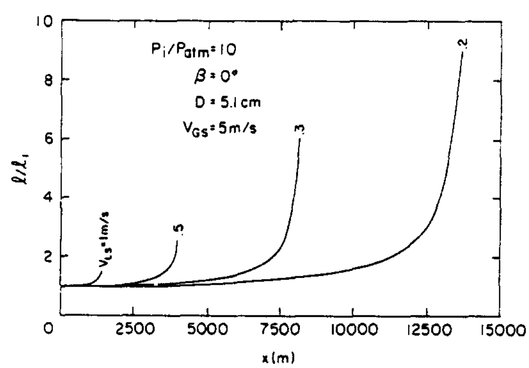

Figure 7. Slug length: effect of gas flow rate.

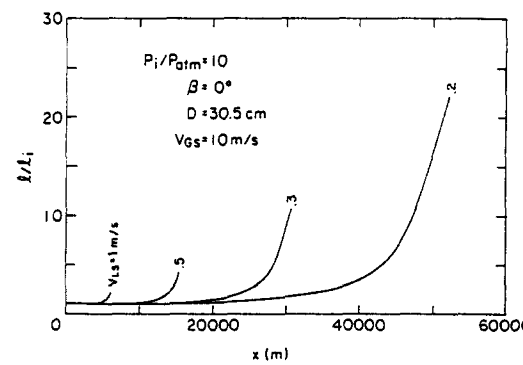

Figure 8. Slug length: effect of pipe diameter.

inclination are small. Note that, contrary to this, the severe slugging phenomenon takes place only for low liquid and gas flow rates and that in the case of high gas flow rate severe slugging does not occur (Taitel 1986).

It was also demonstrated that the long slugs observed in the Prudhoe Bay field test (Brill *et al.* 1981) do not result because of this effect and are probably caused by some kind of unsteady phenomenon, presently under investigation.

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APPENDIX

Equilibrium Film Velocity in Slug Flow

Liquid holdup in the film zone as well as the velocity of the film is calculated as follows.

A liquid continuity balance relative to a moving coordinate system V_t for the slug front that overtakes the liquid film yields

$$R_f(V_t + V_f) = R_s(V_t - V_s), \qquad [A.1]$$

where R_f is the liquid film holdup near the slug front. Usually the solution for R_f and V_f is complicated and requires the solution of the shape of the liquid film level as a function of position.

With a slight loss of accuracy we assume here that V_f is the equilibrium velocity satisfying the momentum balance equation of frictional shear vs gravity:

$$f\frac{\rho_L V_f^2}{2} S_f = \rho_L g A_f \sin\beta. \qquad \text{[A.2]}$$

A_f and S_f are the cross-sectional area of the liquid and the wetted perimeter, respectively. A_f and S_f, as well as R_f, are calculated by

$$A_f = \frac{1}{4}D^2\left[\pi - \cos^{-1}\left(2\frac{h}{D} - 1\right) + \left(2\frac{h}{D} - 1\right)\sqrt{1 - \left(2\frac{h}{D} - 1\right)^2}\right], \qquad \text{[A.3]}$$

$$S_f = D\left[\pi - \cos^{-1}\left(2\frac{h}{D} - 1\right)\right] \qquad \text{[A.4]}$$

and

$$R_f = \frac{A_f}{A}. \qquad \text{[A.5]}$$

V_f and R_f are calculated by the simultaneous numerical solution of [A.1] and [A.2].